

NUMERICAL SOLUTION OF THE BELTRAMI EQUATION

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Abstract. An effective algorithm is presented for solving the Beltrami equation $\bar{\partial}f = \mu \partial f$ in a planar disk. The algorithm involves no evaluation of singular integrals. The strategy, working in concentric rings, is to construct a piecewise linear μ -conformal mapping and then correct the image using a known algorithm for conformal mappings. Numerical examples are provided and the computational complexity is analyzed.

1 Introduction

The Beltrami equation

$$\frac{\partial f(z)/\partial \bar{z}}{\partial f(z)/\partial z} = \mu(z)$$

determines a unique normalized quasiconformal self mapping f of the unit disk $\mathbb{D} = \{z : |z| < 1\}$ in the complex plane. Here μ is a given measurable function in \mathbb{D} with $\|\mu\|_\infty < 1$, and is called the Beltrami derivative (or complex dilatation) of f . We say that f is μ -conformal.

The Beltrami equation has been studied intensively in large measure due to its importance in the theory of deformations of Kleinian groups and their applications to Teichmüller spaces [15],[20]. Other applications of the Beltrami equation are mentioned in the introduction to [5]. Some more recent

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applications, such as mapping of the cerebral cortex use the Beltrami equation in the spirit of its original application, dating back to Gauss, of finding a conformal mapping from a surface in 3-space onto a planar region; this is done effectively in [2] without explicit use of the Beltrami derivative μ .

With the increasing use computer studies it has become of great interest to solve the Beltrami equation numerically. One method for doing this is suggested naturally by the classical existence proof given by Mori-Boyarskii-Ahlfors-Bers [15],[20]. For this method one must evaluate singular integrals of the form

$$T_m g(z) = \iint_{\mathbb{D}} \frac{g(\zeta)}{(\zeta - z)^m} d\xi d\eta, \quad m = 1, 2$$

(defined as Cauchy Principal Values when $m = 2$), and calculate sums of Newmann series of the form

$$\sum \mu T_2(\mu T_2(\dots(\mu T_2(1))).$$

A related approach involving the singular integrals was developed by Daripa and Mashat [4]. Instead of summing the Newmann series, their method involves iteration towards a solution of a related Dirichlet problem. Their work includes refinement of the technique of evaluation of the singular integrals via FFT, which is of interest in itself.

G. B. Williams [32] proposed an alternative method of solving the Beltrami equation, based on conformal welding of circle packings. A sequence of circle packings is produced whose corresponding affine mapping converge to the μ -conformal mapping. The emphasis in [32] is on investigating properties of circle packings and on proving they can be constructed to converge to a solution of the Beltrami equation, and little information is given towards the point of view of numerical analysis.

The present approach to the numerical solution of the Beltrami equation is in the spirit of [32], but with the aim of applying conformal mapping methods other than just circle packing. We start from the observation that the general solution of the Beltrami equation for constant μ , $|\mu| < 1$, is an affine linear mapping of the form

$$f(z) = a(z + \mu \bar{z}) + b$$

for complex constants a, b with $a \neq 0$. Thus general quasiconformal mappings are approximated locally by such real linear affine maps. The essential idea

of our method is as follows. In each element of a mesh division of \mathbb{D} , a first attempt to construct f may be carried out by using such affine maps. Suppose that the piecewise linear μ -conformal mapping f_0 which results from combining these local mappings is continuous and in fact one-to-one from \mathbb{D} to a domain $D = f_0(\mathbb{D})$. Let $h : D \rightarrow \mathbb{D}$ be a conformal mapping, the existence being guaranteed by the Riemann mapping theorem. Then the composition $f = h \circ f_0$ is a μ -conformal self mapping of \mathbb{D} . For numerical work one must usually work with Beltrami derivatives which are continuous or at least piecewise continuous.

The problem of calculating a μ -conformal self-mapping of the sphere S^2 can also be solved numerically via the Disk Algorithm, by using domain and image disks covering all but an insignificant part of the Riemann sphere. Consider also the problem of constructing a conformal mapping from a surface in Euclidean space \mathbb{R}^3 to \mathbb{D} . This can be accomplished by first transforming it nonconformally to a disk in \mathbb{R}^2 and then applying the Disk Algorithm to the induced Beltrami equation.

In Section 2 we gather the basic facts we will need about quasiconformal mappings. In Section 3 we define the Disk Algorithm for solving the Beltrami equation, and prove that it converges to the desired function when the parameters are chosen appropriately. In Section 4 we discuss the choice of conformal mapping algorithm to be plugged into the Disk Algorithm, and the corresponding computational cost. Numerical examples are provided in Section 5.

2 Preliminaries

First we discuss affine linear quasiconformal mappings. Let μ, a, b be complex constants, $a \neq 0$, $|\mu| < 1$, and consider the mappings

$$L[\mu](z) = z + \mu\bar{z} \tag{1}$$

$$A[a, b](z) = az + b \tag{2}$$

for $z \in \mathbb{C}$. Thus $L[\mu]$ is μ -conformal and linear, while $A[a, b]$ is conformal and affine linear. Further, the images of two points, $w_i = A[a, b](z_i)$, $i = 1, 2$, determine the coefficients a and b . All μ -conformal affine linear mappings are of the form $A[a, b] \circ L[\mu]$, and this decomposition is unique. In particular we have the following.

Proposition 1 *Given z_1, z_2 distinct and w_1, w_2 distinct, together with $|\mu| < 1$, there is a unique μ -conformal affine linear mapping $B[\mu; z_1, z_2; w_1, w_2]$ sending z_1 to w_1 and z_2 to w_2 .*

We will be interested in mappings of triangles, which can be described as follows.

Proposition 2 *Given z_1, z_2, z_3 noncollinear and w_1, w_2, w_3 noncollinear, there is a unique affine linear mapping $T[z_1, z_2, z_3; w_1, w_2, w_3]$ which sends z_i to w_i ($i = 1, 2, 3$). Its Beltrami derivative is equal to*

$$-\frac{(z_2 - z_1)(w_3 - w_1) - (z_3 - z_1)(w_2 - w_1)}{(\bar{z}_2 - \bar{z}_1)(w_3 - w_1) - (\bar{z}_3 - \bar{z}_1)(w_2 - w_1)}. \quad (3)$$

Proof. The map $T = T[z_1, z_2, z_3; w_1, w_2, w_3]$ is well-defined because each of the pairs $(z_2 - z_1, z_3 - z_1)$, $(w_2 - w_1, w_3 - w_1)$ is a basis of plane over the field of real numbers. To verify that the Beltrami derivative of T is (3), first consider the function

$$f(\zeta) = T[0, 1, z; 0, 1, w](\zeta) = \frac{1}{z - \bar{z}} \left((w - \bar{z})\zeta + (w - z)\bar{\zeta} \right)$$

for z, w fixed. The Beltrami derivative of f is $\mu = -(w - \bar{z})/(w - z)$. The conformal linear map $h_1 = A[1/(z_2 - z_1), -z_1/(z_2 - z_1)]$ takes z_1, z_2, z_3 to $0, 1, z$ where $z = (z_3 - z_1)/(z_2 - z_1)$. An analogous conformal linear mapping h_2 takes w_1, w_2, w_3 to $0, 1, w$ when w is chosen appropriately. Then A is the composition $h_2^{-1} \circ f \circ h_1$ and its Beltrami derivative

$$(\mu \circ h_1) \frac{\bar{h}'_1}{h'_1}$$

as obtained from the Chain Rule, is seen to be equal to (3). \square

The conformal mapping (2) sends triangles to similar triangles. A Beltrami derivative (3) can be regarded as a measure of how much the triangles with vertices z_1, z_2, z_3 and w_1, w_2, w_3 fail to be similar. This observation will be used in the proof of the following technical result, which in turn will be used in the proof of convergence of the Disk Algorithm.

Lemma 2.1 Fix $z_0 = r_0 e^{i\theta}$ with $r_0 > 0$, and fix μ_0 , $|\mu_0| < 1$. Let $\epsilon > 0$. Then there exists $\delta > 0$ such that the following holds. Suppose $r_1 > r_0$ and $\theta_0 \leq \theta'_0 < \theta_1 \leq \theta'_1 < \theta_2$, and write

$$z_k = r_0 e^{i\theta_k}, \quad z'_k = r_1 e^{i\theta'_k}, \quad k = 1, 2.$$

Further, suppose that $|w_0| = |w_1| = |w_2| = r > 0$ and $|\mu_1|, |\mu_2| < 1$, and set

$$\begin{aligned} w'_1 &= B[\mu_1; z_0, z_1; w_0, w_1](z'_1), \\ w'_2 &= B[\mu_2; z_1, z_2; w_1, w_2](z'_2). \end{aligned}$$

Suppose that z_1, z_2, z'_1, z'_2 are within δ of z_0 , that μ_1, μ_2 are within δ of μ_0 , and that the ratios

$$\frac{\theta_2 - \theta_1}{\theta_1 - \theta_0}, \quad \frac{\theta'_2 - \theta'_1}{\theta'_1 - \theta'_0}, \quad \frac{w_2 - w_1}{w_1 - w_0} \quad (4)$$

are within δ of 1. Let μ be the Beltrami derivative of the affine linear mapping $T[z'_1, z_1, z_0; w'_1, w_1, w_0]$. Then

$$|\mu - \mu_0| < \epsilon.$$

Proof. The positions of the points involved are represented schematically in Figure 1. First suppose that z_0, z_1, z_2 , rather than lying on the circle $|z| = r_0$, are instead collinear and further are equally spaced, and likewise for w_0, w_1, w_2 ; also suppose that the segment z'_1, z'_2 is parallel to z_0, z_1, z_2 and that $|z'_2 - z'_1| = |z_2 - z_1|$. Suppose further that $\mu_1 = \mu_2 = \mu_0$. One may calculate

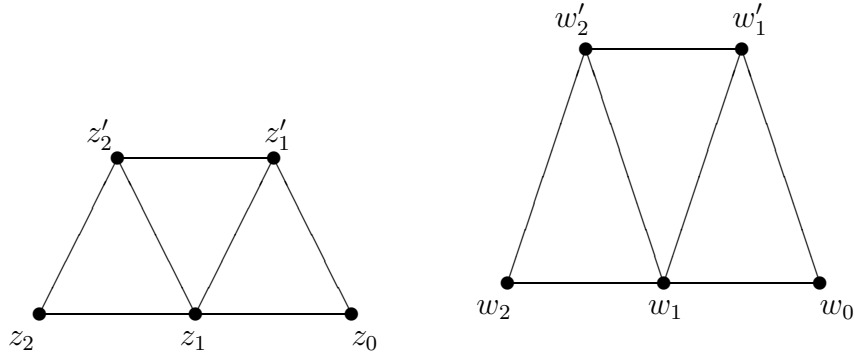


Figure 1: Affine mappings of triangles

from Proposition 2 that the Beltrami differential of $T[z'_1, z_1, z_0; w'_1, w_1, w_0]$ must be equal to μ_0 . Alternatively, one may simply observe that the triangles w'_1, w_1, w_0 and w'_2, w_2, w_1 are similar because the triangles z'_1, z_1, z_0 and z'_2, z_2, z_1 are similar and have parallel bases. Then by elementary geometry the triangle w_1, w'_1, w'_2 is similar to w'_1, w_1, w_0 . Since z_1, z'_1, z'_2 is similar to z'_1, z_1, z_0 , it follows again that the Beltrami derivative of the affine mapping is μ_0 .

Returning now to the hypothesis that $|z_k| = r_0$, by taking δ sufficiently small, which implies that r_1 is close to r_0 , one can assure that the arcs of the circles of radii r_0, r_1 containing the z_k, z'_k will be approximated by straight lines, so the Beltrami derivative μ^* of the affine mapping between the middle triangles can be made to satisfy $|\mu^* - \mu_0| < \epsilon/2$. Up to now we have referred only to equally spaced points, that is, when the ratios (4) are all equal to 1. In general, since μ is a continuous function of $r_1, \theta_1, \theta_2, \theta'_1, \theta'_2, \mu_1$, and μ_2 , we see that by taking δ sufficiently small we can assure that $|\mu - \mu^*| < \epsilon/2$. This completes the proof. \square

We recall some well known properties of quasiconformal mappings [1], [20], [21]. If f_1, f_2 are quasiconformal mappings in the same planar domain and have the same Beltrami derivative, then $f_2 = h \circ f_1$ where h is a conformal mapping from the image of f_1 to the image of f_2 . If μ is measurable in $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ and $\|\mu\|_\infty < 1$, then there is a unique μ -conformal mapping $f: \mathbb{D} \rightarrow \mathbb{D}$ satisfying the normalization

$$f(0) = 0, \quad f(1) = 1.$$

In this paper all mappings will be piecewise smooth.

3 Disk Algorithm for Solving Beltrami Equations

In this section we will describe the main algorithm for solving the Beltrami equation. By assumption a formula or program is provided for calculating the proposed Beltrami derivative $\mu(z)$ in all of \mathbb{D} .

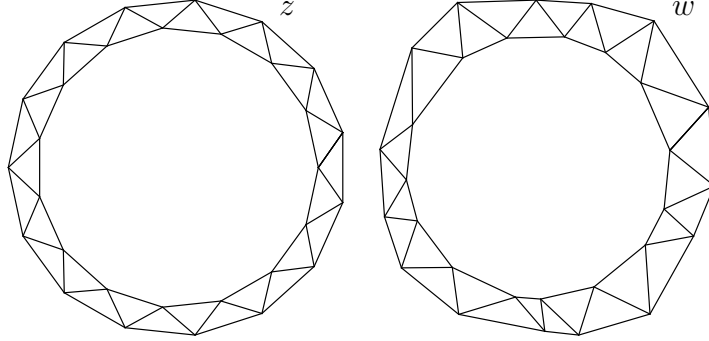


Figure 2: Elements of the Ring Extension.

3.1 Ring Extension procedure

First we describe the basic step. Let z_1, z_2, \dots, z_N be the vertices of a positively oriented simple closed polygon, lying within another such polygon whose vertices are z'_1, z'_2, \dots, z'_N . We suppose that the interiors of the triangles D_k with vertices z_k, z_{k+1}, z'_{k+1} ($k = 1, \dots, N$) are disjoint (indices are taken modulo N). Let $|\mu(z)| < 1$ in the doubly connected domain R between these two polygons. Let w_1, w_2, \dots, w_N also form a simple closed polygon in \mathbb{C} .

The Ring Extension produces w'_1, w'_2, \dots, w'_N as follows. For each k let μ_k be the average value of μ over the triangle D_k ; this exists because prescribed Beltrami differentials are by hypothesis integrable. Define

$$w'_k = B[\mu_k; z_k, z_{k+1}; w_k, w_{k+1}](z'_k), \quad k = 1, \dots, N, \quad (5)$$

where B is as given in Proposition 1. We will say that the Ring Extension is *applicable* for the given data if the interiors of the triangles w_k, w_{k+1}, w'_{k+1} are disjoint.

3.2 Disk algorithm

We now define the main iteration of the algorithm for solving the Beltrami equation in a disk.

The unit disk \mathbb{D} is subdivided by a mesh with vertices z_{jk} , where for each

j , the points $z_{j1}, z_{j2}, \dots, z_{jN}$ form a positively oriented polygon, with

$$|z_{jk}| = r_j, \quad j = 1, \dots, M, \quad 0 < r_1 < \dots < r_M = 1.$$

The arguments

$$\theta_{jk} = \arg z_{jk}, \quad k = 1, \dots, N,$$

of the vertices are determined modulo 2π . In what follows we will take these arguments to be uniformly spaced, though this simplification is not strictly necessary for the basic idea of the algorithm.

Fix j , and assume that the images w_{jk} have been calculated for a μ -conformal mapping $w = f(z)$ from the subdisk $\{|z| < r_j\}$ to $\{|w| < r_j\}$ normalized by $f(0) = 0$. We wish to calculate the images $w_{j+1,k} = f(z_{j+1,k})$.

For convenience of notation, we will suppose that $\theta_{j+1,k}$ is close (modulo 2π) to $(\theta_{j,k} + \theta_{j,k+1})/2$ (recall Figure 1). In all of our applications of the method this will be taken to be exactly so, by using the definition

$$z_{jk} = r_j e^{2\pi i(k+j/2)/N}. \quad (6)$$

If (6) does not hold, then, one must modify slightly the statements and proofs given below. The term $j/2$ in the exponent of (6) has been included in order to relate the indices to Figure 1 and to formula (5). The Disk Algorithm requires carrying out the following for $j = 1, \dots, M$.

DISK ALGORITHM: j -th iteration.

Step 1. Assume that the Ring Extension with Beltrami derivative μ is applicable to the doubly connected region R_j between $z_{j1}, z_{j2}, \dots, z_{jN}$ and $z_{j+1,1}, z_{j+1,2}, \dots, z_{j+1,N}$, mapping the inner points to the vertices $w_{j1}, w_{j2}, \dots, w_{jN}$. Let the resulting points be called

$$\tilde{w}_{j+1,1}, \tilde{w}_{j+1,2}, \dots, \tilde{w}_{j+1,N}.$$

According to (5), these are specifically

$$\tilde{w}_{j+1,k} = B[\mu_{jk}; z_{j,k}, z_{j,k+1}; w_{jk}, w_{j,k+1}](z_{j+1,k}) \quad (7)$$

for $k = 1, \dots, N$.

Step 2. The points $\tilde{w}_{j+1,1}, \tilde{w}_{j+1,2}, \dots, \tilde{w}_{j+1,N}$, joined in order, bound a simply connected polygonal domain D_{j+1} . Let

$$h_{j+1}: D_{j+1} \rightarrow \{|z| < r_{j+1}\}$$

be a conformal mapping satisfying $h_{j+1}(0) = 0$, calculated by any appropriate known method. Define

$$w_{j+1,k} = h_{j+1}(\tilde{w}_{j+1,k}). \quad (8)$$

Step 3. Apply h_{j+1} to each $w_{j',k}$ for $j' \leq j$ and all k . For simplicity of notation we will call the images $w_{j',k}$ as well.

Some details remain to be specified to make this general scheme precise and to complete the Disk Algorithm. One must decide how to start the algorithm by finding the first polygon $w_{11}, w_{12}, \dots, w_{1N}$. We have done this using an explicit formula [29] for the conformal mapping to \mathbb{D} from an ellipse with semimajor and semiminor axes a, b ($a^2 - b^2 = 1$) and foci ± 1 ,

$$w = \sqrt{k} \operatorname{sn} \left(\frac{2K}{\pi} \sin^{-1} u; k^2 \right) \quad (9)$$

where the Jacobi modulus k is related to the complete elliptic integral K and the Jacobi theta functions by

$$q = (a + b)^{-4} = e^{-\frac{\pi K(1-m)}{K(m)}}$$

$$k = \sqrt{m} = \left(\frac{\theta_2}{\theta_3} \right)^2$$

(notation from [31]). Note that the image of the circle $|z| = r_1$ under the mapping $L[\mu]$ is an ellipse with axes $r_1(1 + |\mu|)$, $r_1(1 - |\mu|)$ slanted in the directions $(1/2) \arg \mu$, $(1/2)(\arg \mu + \pi)$ respectively, modulo π . This ellipse is sent by the affine map $A[1/(2r_1\sqrt{\mu}), 0]$ to an ellipse with axes a, b . Then via (9) this is transformed conformally to the unit disk and can then be rescaled to any desired radius, completing the first step of the Disk Algorithm.

A more crucial decision is the choice of conformal mapping procedure in Step 2 of the Disk Algorithm. This will be discussed in Section 4.

3.3 Convergence of Disk Algorithm

Once the Disk Algorithm has been executed, the collection $\{w_{jk}\}_{j,k}$ defines in a natural way a piecewise linear mapping of the polygon $\{z_{Mk}\}_k$ to the polygon $\{w_{Mk}\}_k$, which sends z_{jk} to w_{jk} . If desired, this mapping may be completed to a self mapping of \mathbb{D} by extrapolating to the remaining points near the circumference but for large M this portion of the domain is insignificant. We now justify that this piecewise linear mapping is a valid approximation for the μ -conformal mapping of \mathbb{D} .

For purposes of normalization in the the following theorem, note that $z_{j,-j/2}$ given by (6) is positive real when j is even. We write $[j/2]$ for the greatest integer no greater than $j/2$.

Theorem 2.1 *Let $|\mu(z)| \leq \kappa < 1$ for all $z \in \mathbb{D}$, where μ is piecewise continuous in \mathbb{D} . Let $M_n, N_n \rightarrow \infty$, and for each n , let $0 = r_{n0} < r_{n1} < r_{n2} < \dots < r_{nM_n} = 1$ be chosen in such a way that $\sup_j (r_{nj} - r_{n,j-1}) \rightarrow 0$ as $n \rightarrow \infty$. Suppose further that the Ring Extension is always applicable so that Step 1 can be carried out for all j . Let f_n be the piecewise linear self mapping of \mathbb{D} produced by the Disk Algorithm, normalized so that $w_{j,-[j/2]}$ is positive real. Then $\{f_n\}$ converges uniformly on compact subsets of \mathbb{D} to a μ -conformal self mapping f of \mathbb{D} .*

Proof. Let $\kappa < \kappa' < 1$. By construction, on each outward pointing triangle of the mesh (such as z'_1, z_1, z_0 in Figure 1), f_n is an affine linear quasiconformal mapping whose dilatation $|\mu_n|$ is no greater than κ . Then by Proposition 2.1 we have $|\mu_n| < \kappa'$ on the inward pointing triangles (such as z_1, z'_1, z'_2 in Figure 1) if the mesh is fine enough. Therefore f_n is κ' -quasiconformal for sufficiently large n . By compactness of bounded families of uniformly quasiconformal mappings [21], there is a limit mapping f which is also κ -quasiconformal (it is not constant because $f(0) = 0, f(1) = 1$). Further, μ_n converges uniformly to the Beltrami differential of f almost everywhere, so it follows that f is μ -conformal. \square

3.4 Condition to guarantee applicability

We must now show how to choose radii r_j so that the Ring Algorithm will be applicable in Step 1. The matter is to avoid crossing of images of adjacent

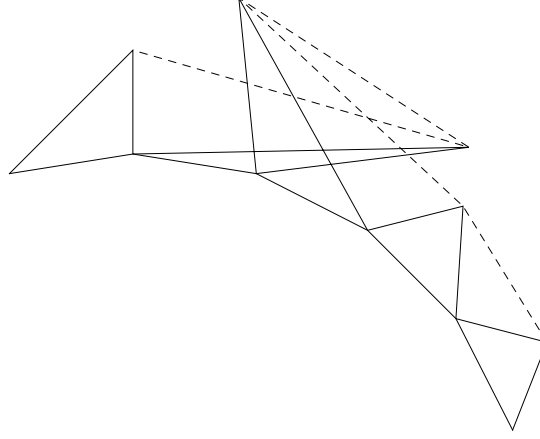


Figure 3: Crossing triangles in the (failed) Ring Extension. The outer polygon (dotted line) is not simple.

triangles as in Figure 3. In the context of Figure 1, let us say that a mapped triangle w_0, w_1, w'_1 is *skewed* when one of the base angles is greater than $\pi/2$, thus, when the perpendicular from the apex w'_1 to the line containing the base w_0, w_1 of the triangle does not pass through the base. It is clear that if none of the triangles mapped in Step 1 is skewed, then the argument of each $\tilde{w}_{j+1,k}$ will lie between those of w_{jk} and $w_{j+1,k}$, so the Ring Extension will be applicable. In general, given N , we will have to take M sufficiently large, taking $\sup |\mu|$ into account as well. This is reflected in the following result.

Theorem 2.2 *Suppose that the inequality*

$$\frac{r_{j+1}}{r_j} < \cos \frac{2\pi}{N} + \frac{(1 - |\mu(z)|)^2}{2|\mu(z)|} \sin \frac{2\pi}{N} \quad (10)$$

is satisfied for all z in the ring $r_j \leq |z| \leq r_{j+1}$. Then all of the N outward-pointing image triangles produced in the j -th application of Step 1 of the algorithm are non-skewed.

Proof. In the context of Figure 1, let us suppose that $z_0 = e^{-i\alpha}z$, $z_1 = e^{i\alpha}z$, and $z'_1 = \beta z$, where $r_j = |z|$, $r_{j+1} = \beta r_j$, and $2\alpha = \theta_1 - \theta_0$. We normalize the figure by subtracting the midpoint $(\cos \alpha)z$ of the segment from z_0 to z_1 , as shown in Figure 4, in which $\beta' = \beta - \cos \alpha$. Carrying out Step 1, we take an average value for μ within the triangle and apply $L[\mu]$. For convenience

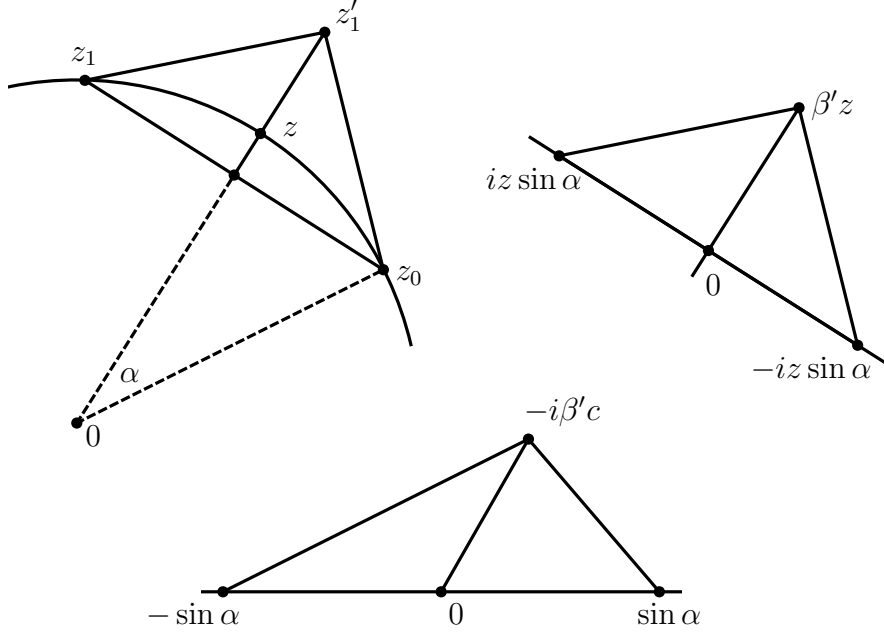


Figure 4: Proof of Theorem 2.2

we follow this by composition with the conformal mapping $A[-i/(z - \mu\bar{z}), 0]$ (recall the formulas (1),(2)). It is thus seen that the condition for the image not to be skewed is

$$|\operatorname{Re} i\beta'c| < \sin \alpha$$

where

$$c = \frac{z + \mu\bar{z}}{z - \mu\bar{z}} = \frac{1 + d}{1 - d}, \quad d = \mu\bar{z}/z.$$

Since $|\operatorname{Re} ic| = |\operatorname{Im} c| = 2|\operatorname{Im} d|/|1 - d|^2 \leq 2|d|/|1 - d|^2$, our condition is satisfied when

$$\frac{\beta'}{\sin \alpha} < \frac{|1 - d|^2}{2|d|}$$

But $\alpha = 2\pi/N$ and $|d| = |\mu|$, so we are done. \square

The condition in this theorem is appropriate for when one is not going to take into account precise information regarding the distribution of $|\mu|$ within \mathbb{D} , and in such cases it is reasonable to make the ratio $\beta = r_{j+1}/r_j$ independent of j . Note that $\beta = 1 + (\pi/N) \tan a$ where a is the base angle of the isosceles mesh triangles. Since $r_M = 1$, the radii are determined as

N	$\mu = 0.1$	$\mu = 0.2$	$\mu = 0.3$	$\mu = 0.4$	$\mu = 0.5$
32	2.561 (5)	1.605 (9)	1.299 (15)	1.156 (27)	1.078 (52)
64	1.789 (8)	1.309 (18)	1.155 (32)	1.083 (58)	1.044 (107)
128	1.396 (16)	1.156 (37)	1.079 (70)	1.043 (127)	1.023 (230)
256	1.198 (34)	1.078 (80)	1.04 (154)	1.022 (279)	1.012 (504)
512	1.099 (71)	1.039 (175)	1.02 (339)	1.011 (614)	1.006 (1108)

Table 1: Ratios r_{j+1}/r_j required by the estimate (10) for given values of N and $\|\mu\|_\infty$. The corresponding values of M prescribed by (11) are shown in parentheses.

$r_j = \beta^{M-j}$. It is also reasonable to take the central “hole” in the mesh not much bigger than the largest (i.e., outer) triangles, which have a base of $2\pi/N$. Since $r_1 = \beta^{1-M}$, this suggests defining M at least as large as

$$M \approx \frac{\log(N/(2\pi))}{\log \beta} \quad (11)$$

if it is desired to assure (10) and to reasonably cover \mathbb{D} with the mesh. (See Table 1.) Of course, larger values of M may be used if a finer mesh is desired.

4 Conformal mapping methods and computational cost of the Disk Algorithm

The vast literature developed during the past century on methods for mapping a simply connected domain conformally to a disk (see [8], [11], [16], [30] and the references therein), gives an ample range of options for implementing Step 2 of the Disk Algorithm.

A natural choice is to apply an osculation method [16], for two reasons: the convergence is guaranteed regardless of the complexity of the original configuration, and most common osculation methods provide an approximation of the mapping from non-circular domain to a disk (rather than the inverse direction). However, it is well known that osculation methods tend to converge slowly.

The fastest available conformal mapping methods, many of which cost $O(N \log N)$ computations per iteration and converge linearly or in some cases quadratically, arrive quickly at good approximations for the boundary maps

$z \mapsto w$, and would be applicable in the Disk Algorithm because the domains involved to be mapped are close to circular. However, such methods provide directly only the boundary values of the conformal mapping, so one must then use integration to calculate each inner value as a boundary integral. Further, it is the inverse mapping $w \mapsto z$ which is needed for the Disk Algorithm. Thus additional programming would be required. Since N may be potentially large, we have ruled out Schwarz-Christoffel methods [3], [10].

In the examples computed in Section 5 we have used the Zipper Algorithm [22]. Briefly, this algorithm approximates a conformal mapping from the interior of an N -point Jordan curve polygon to the unit disk, as the composition of N partial zipper mappings, each of which is in turn a composition of a few Möbius transformations and square roots. To find each partial zipper mapping requires $O(N)$ arithmetic operations, thus giving an operation count of $O(N^2)$ for calculating the zipper mapping. The Zipper Algorithm is similar to the classical osculation mappings in the sense that it provides an explicit formula for the approximated conformal mapping in the interior as well as on the boundary of the domain. It is different in that for fixed N , it provides a single mapping rather than an infinite convergent sequence of mappings.

The calculation of the computational cost of the Disk Algorithm with zipper mappings is as follows. In the j th iteration, Step 1 requires $O(N)$ arithmetic operations. Step 2 requires $O(N^2)$ operations to find the conformal mapping, and in the process produces the N image points of the boundary. Over M values of j , Steps 1 and 2 cost $O(MN^2)$. In Step 3, the mapping, which costs $O(N)$ operations to map a single point, must be applied to j collections of N points each, making $O(jN^2)$ operations. Summing these costs over $1 \leq j \leq M$ gives $O(M^2N^2)$ as the total operation count of Step 3.

The preceding analysis is for the cost applied to the complete triangular mesh in \mathbb{D} . However, there are situations when one may only be interested in calculating $f(z)$ for a reduced subset of interior points; then it is not necessary to apply the conformal mappings to the mesh points other than these. Then the cost of Step 3 may be much lower. For example, with a submesh of M' concentric rings of N' points ($M' < M$, $N' < N$), the calculation count is reduced to $O(MN^2) + O(M'^2N'N)$. As an extreme case, when one only wants to find the boundary map $f|_{\partial\mathbb{D}}$, it is not necessary to keep any of the

inner image data; in other words, Step 3 may be suppressed entirely, and the operation count is then $O(MN^2)$. Examples of proper submeshes (M', N') will be given in Section 5.

Corresponding operation counts may be obtained similarly if other conformal mapping algorithms are used in Step 2 of the Disk Algorithm.

5 Numerical Results

The Disk Algorithm was programmed in Mathematica (Wolfram), on a standard laptop computer of approximately 1GH, with the conformal mapping step being carried out by calling Fortran routines obtained from the website of [22].

The following examples illustrate the results of the Disk Algorithm for a variety of prescribed Beltrami derivatives μ .

(1) *Constant μ .* We take $\mu(z) = 0.4$. The Disk Algorithm was applied for a mesh of equilateral triangles with $(M, N) = (64, 128)$. Figure 5 shows the image of a subdomain with $(M, N) = (32, 32)$.

Note that the image triangles in Figure 5 are not similar, even though μ is constant, since the similarity class of an image triangle depends upon both the value of μ and the slope of the base of the domain triangle.

Note that constant μ doesn't imply that all the triangles are skewed the same way, it depends on the position of the domain triangle too.

(2) *Radial quasiconformal mapping.* Let $\varphi: [0, 1] \rightarrow [0, 1]$ be an increasing diffeomorphism of the unit interval. Then the radially symmetric function

$$f(z) = \varphi(|z|)e^{i \arg z} \quad (12)$$

has Beltrami derivative equal to

$$\mu(z) = \frac{|z|(\varphi'(z)/\varphi(z)) - 1}{|z|(\varphi'(z)/\varphi(z)) + 1} \quad (13)$$

when $z \neq 0$. As an illustration we will take

$$\varphi(r) = 1 + \frac{1}{2} \sin^2 \pi(r - \frac{1}{2})$$

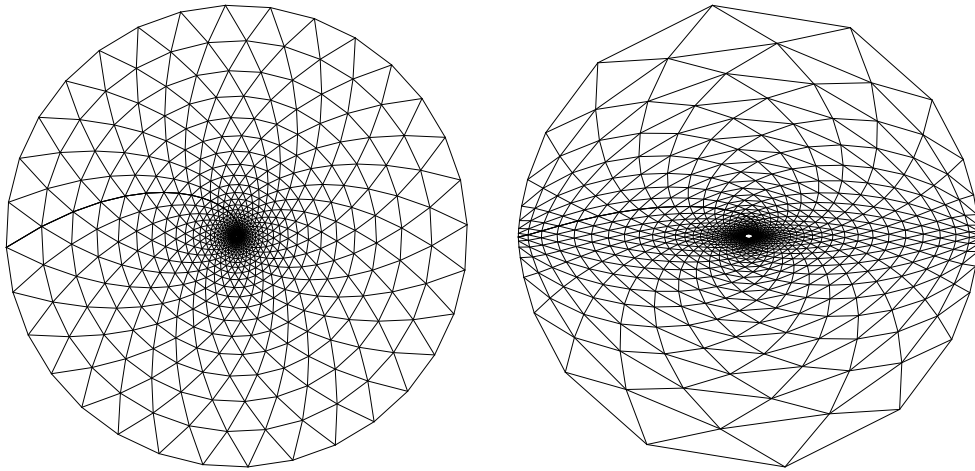


Figure 5: Image for constant Beltrami derivative $\mu = 0.4$. The original triangulation of unit disk is shown at left.

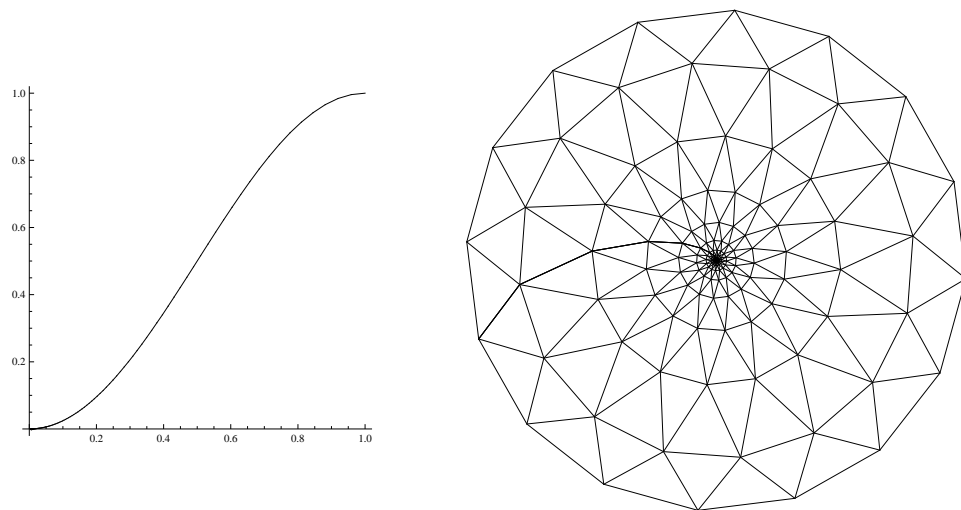


Figure 6: Radial function φ of example 2 (left), together with rotationally symmetric image domain.

as in Figure 6.

The resulting Beltrami derivative satisfies $\|\mu\| = 0.33$ approximately (check). The Disk Algorithm was applied with $(M, N) = (256, 128)$, with only the real values being conserved in Step 3. These real values were compared with the true values $\varphi(r)$ and found to have an error no greater than 0.0025. The image region is depicted in Figure 6.

(3) *Sectorial quasiconformal mapping.* Let $\psi: [0, 2\pi] \rightarrow [0, 2\pi]$ be an increasing diffeomorphism. Write $\tilde{\psi}(e^{i\theta}) = e^{i\psi(\theta)}$. Then the sectorially symmetric function

$$f(z) = |z| \tilde{\psi} \left(\frac{z}{|z|} \right) \quad (14)$$

has Beltrami derivative equal to

$$\mu(z) = \frac{1 - \psi'(\theta)}{1 + \psi'(\theta)} \frac{\bar{z}}{z} \quad (15)$$

when $z \neq 0$. As an example we will take

$$\psi(\theta) = \begin{cases} \frac{\theta}{2}, & 0 \leq \theta \leq \pi, \\ \frac{\pi}{2} + \frac{3(\theta-\pi)}{2}, & \pi \leq \theta \leq 2\pi. \end{cases}$$

The Disk Algorithm was applied with $(M, N) = (256, 128)$, conserving no interior values in Step 3. The final boundary values were compared with the true values $\psi(\theta)$ and found to have an error no greater than 0.055. The image region is depicted in Figure 7.

(4) *Exterior mappings.* In Daripa [5], quasiconformal mappings from \mathbb{D} to the exterior of an ellipse (the origin being sent to ∞) are calculated with the following two sample Beltrami derivatives,

$$\begin{aligned} \mu_1(z) &= |z|^2 e^{0.65(iz^5 - 2.0)}, \\ \mu_2(z) &= \frac{1}{2} |z|^2 \sin(5 \operatorname{Re} z). \end{aligned}$$

Daripa uses M radii equally spaced in $[0, 1]$, in contrast to the exponential spacing we have been using. The exterior mapping results can be related to those of the Disk Algorithm by use of the formula

$$h(z) = \frac{(1 + \alpha) - (1 - \alpha)z^2}{2\alpha z}$$

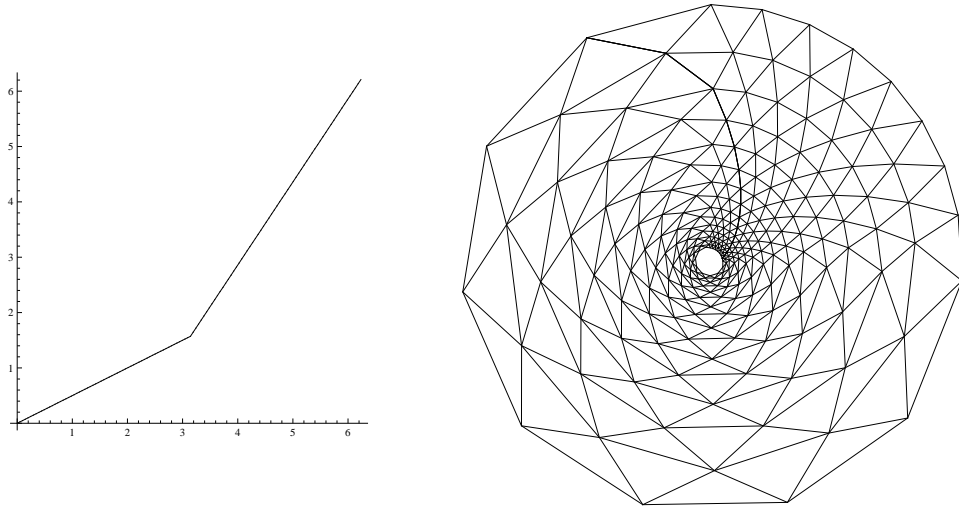


Figure 7: Angular function ψ of example (3) (left), together with image domain under sectorial mapping.

which transforms \mathbb{D} conformally to the exterior of an ellipse with aspect ratio α . Composition of h following the quasiconformal self mapping of D provides a mapping to the exterior of the ellipse with the same Beltrami derivative.

To match the examples in [5], $\alpha = 0.6$ is specified. (However, the inner ellipses in [5] appear to have aspect ratios of approximately 0.47; axes are not drawn.) Our results are depicted in Figure 8. The images we obtain with $(M, N) = (64, 64)$ and $(M, N) = (256, 256)$ turn out to be indistinguishable. These images appear a bit different from the figures shown in [5], and it is not clear that the discrepancies can be accounted for simply by a different selection of level curves.

Computation times are reported in [5] for $N = 64$ as approximately 8.5 seconds of CPU on a MIPS computer described as “approximately 15 times slower than the CRAY-YMP at Texas A & M University.” Our laptop CPU times, using Mathematica/Fortran, were approximately 85 seconds.

(5) *Quasiconformal deformation of Fuchsian groups.* In this example μ is taken to be a quadratic differential for a Fuchsian group G (see [20] for definitions). The elliptic modular group is the collection of conformal transformations of the upper half plane generated by the two mappings $\tau \mapsto \tau + 1$, $\tau \mapsto -1/\tau$. This Fuchsian group has a standard fundamen-

tal domain $\{\tau : \text{Im } \tau > 0, |\text{Re } \tau| < 1/2, |\tau| > 1\}$. The Klein modular function J satisfies the invariance relations $J(\tau) = J(\tau + 1) = J(1/\tau)$, from which it follows that

$$J'(\gamma(\tau))\gamma'(\tau) = J'(\tau)$$

for every element τ in the elliptic modular group. The upper half plane is transformed into the unit disk \mathbb{D} via the elementary conformal mapping

$$z = h(\tau) = i \frac{1 - \tau}{1 + \tau}.$$

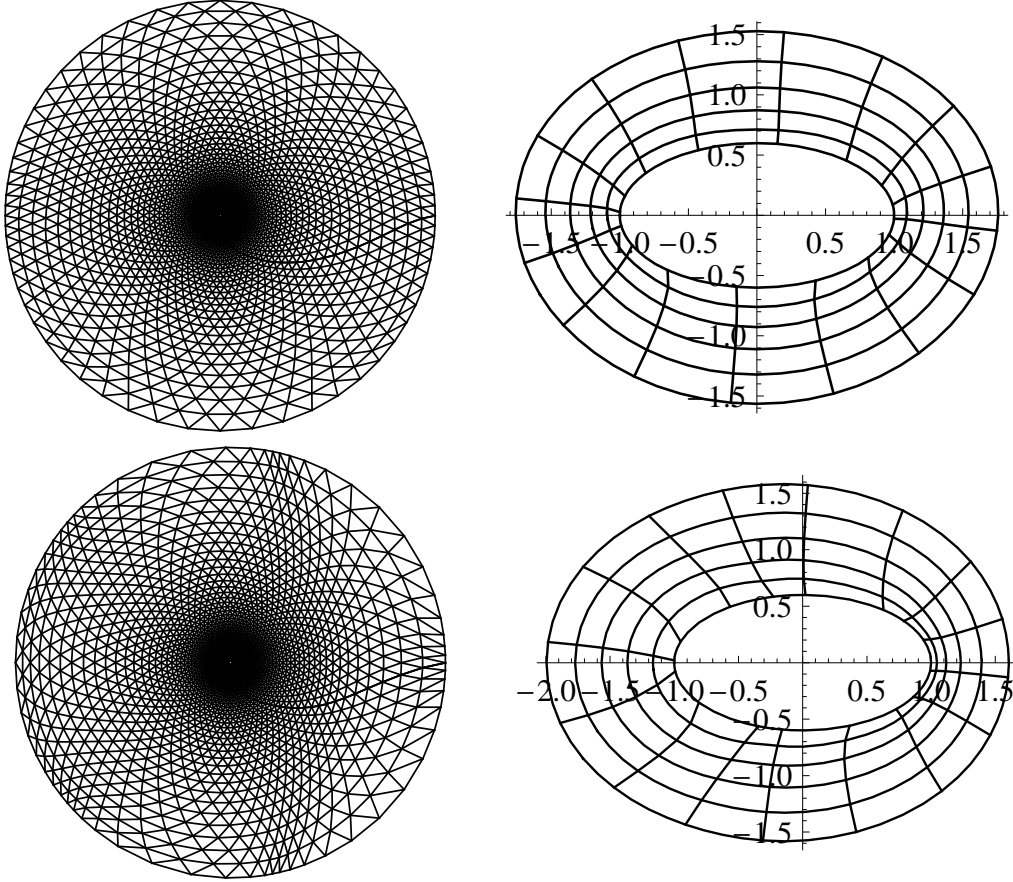


Figure 8: Self mapping of unit disk with Beltrami derivative μ_1 (left), and image of grid near unit circle under exterior conformal mapping for ellipse. Above: Beltrami derivative μ_1 ; below: Beltrami derivative μ_2 .

Let G denote the corresponding group of self mappings of \mathbb{D} . Let $\varphi(z) = J(\tau)$. Then for $0 < \alpha < 1$ the function

$$\mu(z) = \alpha \frac{\overline{\varphi'(z)}}{\varphi'(z)}$$

is a Beltrami differential for the group G ; that is, $\mu(\gamma(z))\overline{\gamma'(z)}/\gamma'(z) = \mu(z)$ for every γ in G . This implies that any μ -conformal self mapping $f: \mathbb{D} \rightarrow \mathbb{D}$ satisfies $f(\gamma(z)) = \gamma_\mu(f(z))$ where $\gamma \mapsto \gamma_\mu$ is an isomorphism from G to another Fuchsian group G_μ .

We have taken $\alpha = 0.4$. Figure 9 shows part of the tessellation of \mathbb{D} by fundamental domains of G and their images, which are fundamental domains of G_μ . Some inaccuracies are clearly visible inasmuch as the image curves must be hyperbolic geodesics.

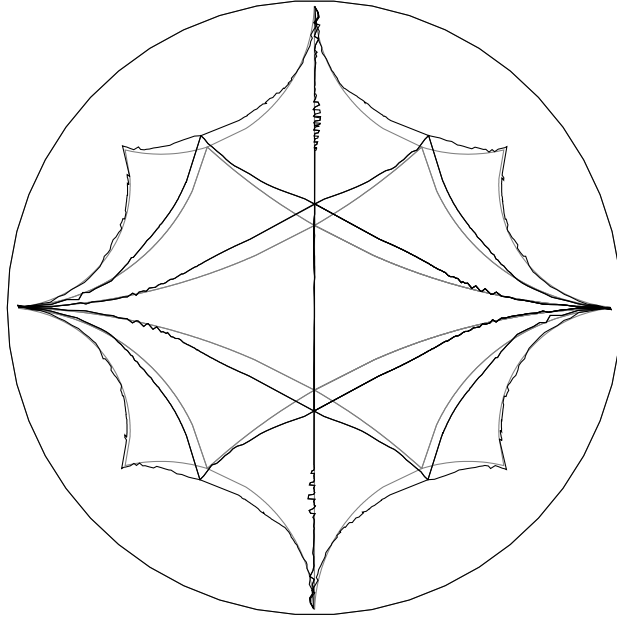


Figure 9: Self mapping of unit disk with Beltrami derivative defined by the Klein modular function. Computations were made with $(M, N) = (256, 512)$, $(M', N') = (128, 64)$. The standard fundamental domain in the upper half plane has been mapped to the unit disk (lighter contour); the image under the quasiconformal mapping is superimposed.

6 Discussion and Conclusions

It is stated in Daripa [5] that prior to that article there were no constructive methods published for solving the Beltrami equation numerically. We discuss here some aspects of [5] in relation to the Disk Algorithm.

As we mentioned in the Introduction, the method of [5] is based on evaluation of singular integrals. That method is presented in the context of finding a μ -conformal mapping to a prescribed star-shaped domain, and is in some ways reminiscent of the classical method of Theodorsen [16] for conformal mappings. Convergence proofs are not supplied.

Daripa's main algorithm requires evaluation of the $\partial/\partial\bar{z}$ derivatives which appear in the singular integrals. A variant is also proposed which does not require these derivatives; however this is not used in the numerical examples provided. The operation count of one iteration of Daripa's method is $O(MN \log N)$. This should be multiplied by the average number of iterations required, which depends on how refined the mesh is and how much accuracy is desired.

The examples we have given in Section 5 show that our method begins to have difficulties when $|\mu|$ is large as 0.5. Likewise, in the examples which we have taken from [5], $\|\mu\|_\infty$ is approximately 0.5, but it should be noted that $|\mu(z)|$ is in fact bounded by 0.12 for $|z| < 0.5$, and by 0.05 for $|z| < 0.3$. In fact, an important limitation stated in [5] is that the Beltrami derivative μ must be Hölder continuous. Further, it is recommended that μ vanish at least as fast as $|z|^3$ at the origin. The Disk Algorithm, in contrast, is not subject to any continuity requirement on μ .

We believe that the Disk Algorithm for solving the Beltrami equation is conceptually simpler than other methods which have been presented, and is easier to implement.

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